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Discussion on the 6th Problem of the 29th International Mathematical Olympiad

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Abstract

With respect to the 6th problem of the 29th IMO, let $b = a + n$ for the equation of the problem, it gets an equation about a, using the mathematical theory to solve the equation about a, because $a > 0, b > 0$, in order to obtain a positive root a, the calculation results in $k < n^2$, it demonstrates that $k \neq n^2$. Moreover, from $b = a + n$, if a and b take all of the positive integers, n will also take all of the positive or negative integers. Therefore, the paper proves that there are no positive integers $a > 0, b > 0$ and $k > 0$ such that $\frac{a^2+b^2}{ab+1} = k$ is the square of an integer except the case of $a = b = 1$.

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Introduction

The 29th international mathematical Olympiad (IMO) held in Australia, although some of the attendants achieved the gold medal, no one solved the 6th problem of this competition,[1] the committee of the organization did also not solve this problem within 6 hours, besides, the four great experts in the number theory in Australia were also unable to solve this problem. Many people think that the 6th problem of the 29th IMO is the most difficult problem in the history of the IMO. Therefore, I think that this problem is very strange, and it is valuable for our study and investigation.[2] This paper will present some discussion on the 6th problem of the 29th IMO.

Materials and Method

The 6th problem of the 29th IMO is described as the following:[1]

Let a and b be positive integers such that ab+1 divides $a^2 + b^2$, show that $\frac{a^2+b^2}{ab+1} = k$ is the square of an integer.

Solution

At first, it is not difficult to find that when $a = b = 1$, $\frac{a^2+b^2}{ab+1} = 1 = 1^2$, it is in concordance with the conclusion of the problem. But if $a \neq b \neq 1$, it is very difficult to

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find a group of positive integers a,b,k such that $\frac{a^2+b^2}{ab+1}=k$ is the square of an integer. Therefore, it is reasonable to consider that there are no positive integers a,b,k, such that $\frac{a^2+b^2}{ab+1}=k$ is the square of an integer.

In order to conveniently discuss the equation $\frac{a^2+b^2}{ab+1}=k$,

let $b=a+n$, here n is an arbitrary integer, in order to make b be positive, the integer $n \ni [-a+1,+\infty)$, substituting it into $\frac{a^2+b^2}{ab+1}=k$, it arrives

$$(2-k)a^2+n(2-k)a+(n^2-k)=0 \quad ...(1)$$

this is an equation about a, eq. (1) is equivalent of

$$(k-2)a^2+n(k-2)a+(k-n^2)=0 \quad ...(2)$$

solving eq. (2) it obtains

$$a=\frac{-n(k-2)\pm\sqrt{n^2(k-2)^2-4(k-2)(k-n^2)}}{2(k-2)} \quad ...(3)$$

In consideration of eq. (2), and in general, $k>2$, to get a positive root a, it is required that

$$\sqrt{n^2(k-2)^2-4(k-2)(k-n^2)}>|-n(k-2)| \quad ...(4)$$

therefore,

$$(\sqrt{n^2(k-2)^2-4(k-2)(k-n^2)})^2>|-n(k-2)|^2 \quad ...(5)$$

eq. (5) can be simplified as

$$-4(k-2)(k-n^2)>0 \quad ...(6)$$

it results in $n^2>k$ ...(7)

Result and Discussion

Using the algebraic equation and the theory of inequality, the paper rigorously obtains eq. (7): $n^2>k$. Eq.(7) demonstrates that $k\neq n^2$, but the integer $n \ni [-a+1,+\infty)$, *if* $a,b \ni [+1,+\infty)$,, from $b=a+n$, n can be an arbitrary integer which belongs to $(-\infty,+\infty)$, for example, if a=1000, n can take the value: -1000+1= -999, -1000+2= -998,…….0, 1, 2,……998, 999, 1000,……; thus, b can be 1, 2, 3, ……1000,……., it points out that $n\neq 0$, because if $n=0$, from eq. (7) and $\frac{a^2+b^2}{ab+1}=k$, it results in that $a^2=b^2<0$, eq.(7) proves that k cannot be just the square of an integer, the conclusion of this problem is impossible except the special case of $a=b=1$. Therefore, that the so-called the most difficult and challenging problem in the history of IMO is in fact impossible, I think that this is the cause of that why no one over the world can solve this problem.

Conclusion

To solve the 6th problem of the 29th IMO, at first, let $b=a+n$, the integer $n\in[-a+1,+\infty)$. However, if $a,b\in[1,+\infty)$, from $b=a+n$, n can take the value of all the positive or negative integers. Substituting $b=a+n$ into $\frac{a^2+b^2}{ab+1}=k$, it gets an equation of a, namely eq. (1), because $a>0$, in order to obtain a positive root a, it is required in eq.(3) that $\sqrt{n^2(k-2)^2-4(k-2)(k-n^2)}>|-n(k-2)|$, this results in $n^2>k$, it demonstrates that corresponding to any integer n, from $\frac{a^2+b^2}{ab+1}=k$ it gets $n^2>k$, hence, $k\neq n^2$. But n can take the value of any positive or negative integer, therefore, there are no positive integers a, b and k such that $\frac{a^2+b^2}{ab+1}=k$ is the square of an integer except the case of $a=b=1$.

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The sole author was responsible for the conceptualization, methodology, data collection, analysis, writing, and final approval of the manuscript.

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