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Binary Third degree Diophantine Equation $5(x-y)^3= 8xy$

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Abstract

This article emphasizes on finding non-zero different integer solutions to binary third degree Diophantine equation $5(x-y)^3 = 8xy$. Two different sets of solutions in integers are presented. Some fascinating relations from the solutions are obtained. The method to get second order Ramanujan numbers is illustrated.

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Introduction

The third degree Diophantine equations are enormous in variety and they have contributed to expansion of research in this field.[1,2] For an extensive approach of these types of problems, one may refer.[3-28] In this article a search is made to get solutions in integers for the considered problem through employing linear transformations. Also, the method of getting second order Ramanujan numbers from the obtained solution is discussed. Some fascinating relations from the solutions are presented.

Method of Analysis

The non-homogeneous binary third degree equation under consideration

$$5(x-y)^3 = 8xy \quad \ldots(1)$$

The substitution of the linear transformations

$$x = u + kv, \; y = u - kv, \; u \neq kv \neq 0, \; k > 0 \quad \ldots(2)$$

in (1) leads to

$$u^2 = k^2 v^2 (1 + 5kv) \quad \ldots(3)$$

Let

$$\alpha^2 = 1 + 5kv \quad \ldots(4)$$

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which, after some calculations, is satisfied by

$$v = v_0 = 5kn^2 + 2n, \alpha = \alpha_0 = 5kn + 1 \quad ...(5)$$

Assume the second solution to (4) as

$$v_1 = v_0 + 2\alpha_0 + 5k, \alpha_1 = \alpha_0 + 5k \quad ...(6)$$

where h is an unknown to be determined. Substituting (6) in (4) and simplifying, we have

$$h = 2\alpha_0 + 5k$$

and in view of (6), it is seen that

The repetition of the above process leads to the general solution to (4) as

$$v_1 = v_0 + 2\alpha_0 + 5k, \alpha_1 = \alpha_0 + 5k \quad ...(7)$$

From (3), we have

$$u_N = k v_N * \alpha_N$$
$$= k (v_0 + 2N\alpha_0 + 5kN^2)(\alpha_0 + 5kN) \quad ...(8)$$

In view of (2), we have

$$x_N = u_N + k v_N = k v_N (\alpha_N + 1)$$
$$= k (v_0 + 2N\alpha_0 + 5kN^2)(\alpha_0 + 5kN + 1),$$
$$y_N = u_N - k v_N = k v_N (\alpha_N - 1) \quad ...(9)$$
$$= k (v_0 + 2N\alpha_0 + 5kN^2)(\alpha_0 + 5kN - 1)$$

Thus, (1) is satisfied by (9) .
To obtain the relations among the solutions, one has to go for taking particular values to the parameters. For simplicity and brevity, we consider the integer solutions to (1) taking

$$k = 1, N = 0, v_0 = 5n^2 + 2n, \alpha_0 = 5n + 1$$

in (9) and they are given by

A few numerical values for the obtained solutions (9) to equation (1) are presented in Table 1 below:

$$x_0 = x(n) = (5n^2 + 2n)(5n + 2),$$
$$y_0 = y(n) = (5n^2 + 2n)(5n)$$

Table 1-Numerical values

1	49	35
2	288	240
3	3*289	17*45
4	4*484	22*80
5	5*729	27*125

From the above Table 1, it is seen that both the values of are alternatively odd and even.

A few interesting relations among the solutions are presented below:

1. $5[2y(k) - x(k) + 4k]$ is a cubical integer
2. $k(2y(k) - x(k))$ is written as difference of two squares
3. $k(2x(k) - y(k))$ is written as difference of two squares
4. $x(k) - y(k) - 2Ct_{6,k} + 2k + 2$ is a perfect square
5. $x(k) - y(k) - 13k = t_{22,k}$
6. $\sum_{k=1}^{n}[x(k) - y(k)] = \frac{20P_n^5 + 22t_{3,n}}{3}$
7. $\sum_{k=1}^{n} y(k) = \frac{20t_{3,n} + 189P_n^5 + t_{3,n} * t_{152,n}}{6}$
8. $25xy$ is a cubical integer
9. $25k^3 x(k) = (y(k))^2$
10. $x(2^n) - y(2^n) = Th_{2n} + M_{n+2} + 7M_{2n} + 9$
11. $x(k) - y(k)$ is a perfect square when takes the values

$$k = k_n = \frac{(-1)^n \beta_{n+1} - 2}{10}, n = -1, 0, 1, 2, ...$$

where

$$\beta_{n+1} = 19f_n + 6\sqrt{10}g_n,$$
$$f_n = (19 + 6\sqrt{10})^{n+1} + (19 - 6\sqrt{10})^{n+1},$$
$$g_n = (19 + 6\sqrt{10})^{n+1} - (19 - 6\sqrt{10})^{n+1}$$

12. $[x(k+2) - y(k+2)] - 2[x(k+1) - y(k+1)] + [x(k) - y(k)] = 20$
$[x(k+4) - y(k+4)] - 2[x(k+3) - y(k+3)] + 2[x(k+2) - y(k+2)]$

13. –2 [x(k+1) y(k+1)]+ [x(k) – y(k)]=40

14. [x(n+k) – y(n+k)] – x(n+k –1) –y(n+k –1)]= 20(k+n) –6
15. $y(k)=20P^5_k + 6\,CP^{15}_k + 9\,k$
16. $y(k)=20P^5_k + 3\,CP^{16}_k + 3\,CP^{14}_k + 9\,k$

Formulation of Second Order Ramanujan Numbers :

From each of the solutions of (1) given by (9) , one can find Second order Ramanujan numbers with base numbers as real integers.

Illustration 1

Consider

$$y(k) = 5k^2 (5k+2)$$
$$= 5k^2 *(5k+2) = (5\,k^2 + 2\,k) * 5\,k$$
$$= A * B = C * D \text{ say}$$

From the above relation, one may observe that

$$(A+B)^2 + (C–D)^2 = (A–B)^2 + (C+D)^2 = A^2 + B^2 + C^2 + D^2$$
$$(5\,K^2+5\,K + 2)^2 + (5k^2 – 3k)^2 = (5\,k^2 – 5k –2)^2 + (5\,k^2 – 7k\,)^2$$
$$=50\,k^4 + 20k^3 + 54\,k^2 + 20k + 4$$

Thus, $50\,k^4 + 20k^3 + 54\,k^2 + 20k + 4$ represents the second order Ramanujan number.

Illustration 2

Consider

$$x(k) = k(5k+2)^2$$
$$= (5k^2 +2k) * (5\,k + 2)^2 * k$$
$$A * B = E * F \text{ say}$$

In this case, the corresponding Second order Ramanujan number is found to be

$$650\,k^4 + 1020k^3 + 630\,k^2 + 180k + 20$$

It is worth mentioning that , in addition to the solutions (5), we have an another set of solutions in integers to (4) given by

$$v= v_0 = 5k\,n^2 - 2n,\ \alpha=\alpha_0 =5k\,n-1$$

and taking

$$k=1,\ N = 0,\ v_0 = 5n^2 - 2n,\ \alpha_0 = 5\,n-1$$

in (9), the corresponding integer solutions to (1) are given by

$$x_0 = x(n) = (5n^2 - 2n)\,(5n)$$
$$y_0 = y(n) = (5n - 2)^2\,(n)$$

Conclusion

This article gives an approach to solve third degree equation with two unknowns though different methods to get solutions in integers. The research in this field may attempt to find various other methods to solve binary cubic equation and also approach to get second order Ramanujan numbers and find various other relation from the obtained solution.

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Conflict of interest

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